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ON PERTURBATIONS ASSOCIATED WITH THE FINAL MOMENTUM IN THE PROBLEM OF STRONG POINT EXPLOSION

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The unsteady motion of perfect gas is considered for considerable values of the characteristic time. The unperturbed gas is assumed quiescent at zero pressure and of constant density, while in the perturbed motion region the energy and momentum of gas are assumed constant. In the first approximation the motion is self-similar and corresponds to a strong point explosion. The gasdynamic functions that define the final momentum are of asymmetric form over the space of corrections to the self-similar solution. The use of the momentum integral, which is valid for the linear problem, simplifies the problem analysis.

The problem of motion of perfect gas at sudden release of energy at a point has a self-similar solution which was first obtained in [1 —3]. Considerable attention was given to the linear problems associated with the linearization of equtions of gasdynamics related to the solution of the problem of strong explosion. Thus the entropy integral valid for any approximation was determined in [4 —6] for linear equations. In addition to the entropy integral for linear systems a number of integrals valid for some particular perturbations was obtained in [7, 8]. These integrals were the corolary of some law of conservation valid for equations of gasdynamics [8]. Among the integrals determined in [7] is the momentum integral. A detailed study of solutions containing the momentum integral was carried out in [9, 10] for plain and axisymmetric motions in the first approximation. Derived solutions provided a good interpretation of perturbations downstream of a plane or arbitrary body in a hypersonic stream, associated with the lift acting on the body. Below, a perturbed solution is derived for the basic, centrally symmetric motion. That solution defines the final momentum of gas, which does not vary with time and for which the momentum integral in [8] is valid.

Let us consider a perfect gas with constant specific heats ratio $\varkappa$ $(1 < \varkappa < 2)$. We denote the gas density, pressure, and temperature by ρ, p, and T, respectively, and introduce a spherical system of coordinates r, φ, ϑ. Projections of the velocity vector $\mathbf{V}$ on axes of this system are denoted by corresponding subscripts, as follows: v_r, v_φ, and v_ϑ.

We assume that the unsteady motion of gas is induced by an explosion and that momentum I_z directed along axis z, from which we shall measure angle ϑ, is imparted to the gas in addition to energy. Let the unperturbed gas be cold and quiescent $(T_1 = 0$ and $\mathbf{V}_1 = 0)$. The assumptions about unperturbed gas parameters make it possible to conclude that during the whole motion process in the perturbation region energy E and I_z remain constant at any instant of time t.

The subsequent analysis is restricted to the asymptotic solution for $t \to \infty$. In the absence of momentum $I_z = 0$ the solution of such problem is self-similar [1–3]. According to it a spherical shock wave $r_s = (bt)^{2/5}$ propagates over the unperturbed gas. Let us assume that in the presence of momentum $I_z \neq 0$ the position of the shock wave for $t \to \infty$ may be defined in the form

$$r_s = (bt)^{2/5}\,[1 + t^{-2m/5} R(\varphi, \vartheta) + \ldots], \quad m > 0 \tag{1}$$

where parameter m and function R are selected so as to satisfy the condition $I_z =$ const $\neq 0$. Function $R(\varphi, \vartheta)$ is assumed to be twice differentiable and $R(0, \vartheta) = R(2\pi, \vartheta)$, which implies that it can be expanded in an absolutely and uniformly convergent series

$$R(\varphi, \vartheta) = \sum_{l=0}^{\infty} \sum_{k=0}^{l} a_{lk} P_l^k (\cos\vartheta) \cos(k\varphi + b_{lk})$$

where a_{lk} and b_{lk} are coefficients of the expansion of function R, $P_l^k(\cos\vartheta)$ are associated Legendre functions, and $P_l^k(\cos\vartheta)\cos(k\varphi + b_{lk})$ is a spherical function of the first kind which is the solution of the differential equation in partial derivatives [11]

$$\frac{\partial^2 Y_l^k}{\partial\varphi^2} + \sin\vartheta \frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial Y_l^k}{\partial\vartheta}\right) + l(l+1)\sin^2\vartheta\, Y_l^k = 0 \tag{2}$$

A shock front expansion similar to (1) was used in [12].

Since the problem of perturbations is linear, we take from the spherical function series a term of an arbitrary ordinal number and write $R = Y_l^k(\varphi, \vartheta)$. The parameters of gas that correspond to this perturbation of the shock wave are sought in the form

$$\begin{aligned}
v_r &= \frac{4}{5(\varkappa+1)} b^{2/5} t^{-3/5} [f(\lambda) + t^{-2m/5} f_m(\lambda) Y_l^k + \ldots] \qquad (3)\\
v_\varphi &= -\frac{4}{5(\varkappa+1)} b^{2/5} t^{-3/5-2m/5} u_m(\lambda) \frac{1}{\sin\vartheta} \frac{\partial Y_l^k}{\partial\varphi} + \ldots\\
v_\vartheta &= -\frac{4}{5(\varkappa+1)} b^{2/5} t^{-3/5-2m/5} w_m(\lambda) \frac{\partial Y_l^k}{\partial\vartheta} + \ldots\\
\rho &= \frac{\varkappa+1}{\varkappa-1} \rho_1 [g(\lambda) + t^{-2m/5} g_m(\lambda) Y_l^k + \ldots]\\
p &= \frac{8}{25(\varkappa+1)} \rho_1 b^{4/5} t^{-6/5} [h(\lambda) + t^{-2m/5} h_m(\lambda) Y_l^k + \ldots]\\
\lambda &= r/(bt)^{2/5}
\end{aligned}$$

The first approximation functions f, g, and h specify the self-similar motion that defines a strong point explosion [1–3]. Let us consider the second approximation functions with index m. The Rankine – Hugoniot conditions at the shock wave front (1) make possible the determination of initial conditions at point $\lambda = 1$

$$\begin{aligned}
f_m &= \frac{\varkappa-7}{2(\varkappa+1)} - m, \quad u_m = 1, \quad w_m = 1 \qquad (4)\\
\rho_m &= -\frac{5\varkappa+13}{\varkappa^2-1}, \quad h_m = \frac{1-7\varkappa}{\varkappa^2-1} - 2m
\end{aligned}$$

Substituting expansions (2) into the system of Euler's equations and retaining terms with like powers of t, we obtain a system of five equations for the second approximation functions. But functions u_m and w_m are solutions of the same ordinary differential equation. Taking into account that the initial conditions (4) for functions u_m and w_m are the same, we conclude that $u_m(\lambda) = w_m(\lambda)$.

Retaining subsequently only $w_m(\lambda)$ and allowing for equality (2), we obtain the system of equations

$$
\begin{aligned}
& g\frac{df_m}{d\lambda} + \left(f - \frac{\varkappa+1}{2}\lambda\right)\frac{dg_m}{d\lambda} + \left(\frac{dg}{d\lambda} + \frac{2}{\lambda}g\right)f_m + \\
&\quad \frac{l(l+1)}{\lambda}gw_m + \left[\frac{df}{d\lambda} + \frac{2}{\lambda}f - \frac{m(\varkappa+1)}{2}\right]g_m = 0 \\
& \left(f - \frac{\varkappa+1}{2}\lambda\right)g\frac{df_m}{d\lambda} + \frac{\varkappa-1}{2}\frac{dh_m}{d\lambda} + \left[\frac{df}{d\lambda} - \frac{(3+2m)(\varkappa+1)}{4}\right]gf_m + \\
&\quad \left[\left(f - \frac{\varkappa+1}{2}\lambda\right)\frac{df}{d\lambda} - \frac{3}{4}(\varkappa+1)f\right]g_m = 0 \\
& \left(f - \frac{\varkappa+1}{2}\lambda\right)g\frac{dw_m}{d\lambda} + \left[\frac{f}{\lambda} - \frac{(3+2m)(\varkappa+1)}{4}\right]gw_m - \frac{\varkappa-1}{2\lambda}h_m = 0 \\
& \varkappa h\frac{df_m}{d\lambda} + \left(f - \frac{\varkappa+1}{2}\lambda\right)\frac{dh_m}{d\lambda} + \left(\frac{dh}{d\lambda} + \frac{2\varkappa}{\lambda}h\right)f_m + \\
&\quad \frac{\varkappa l(l+1)}{\lambda}hw_m + \left[\varkappa\frac{df}{d\lambda} + \frac{2\varkappa}{\lambda}f - \frac{(\varkappa+1)(3+m)}{2}\right]h_m = 0
\end{aligned}
\tag{5}
$$

Among the shock front perturbation we consider those that provide the final contribution to the gas momentum along the z-axis when $t \to \infty$. The expression for I_z is of the form

$$I_z = \lim_{r \to 0}\int_0^{2\pi}\int_0^{\pi}\int_r^{r_s} \rho v_z r^2 \sin\vartheta\, dr\, d\vartheta\, d\varphi, \quad v_z = v_r\cos\vartheta - v_\vartheta\sin\vartheta$$

Taking into consideration expansions (3) and the formula for the spherical function $Y_l^k = p_l^k(\cos\vartheta)\cos(k\varphi + b_{kl})$, we obtain

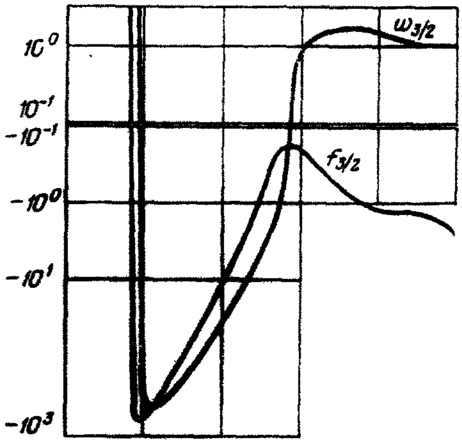
$$
\begin{aligned}
I_z = {} & \frac{4}{5(\varkappa-1)}\rho_1 b^{8/5} t^{8/5 - 2m/5}\int_0^{2\pi}\cos(k\varphi + b_{lk})\,d\varphi\int_0^{\pi}P_l^k(\cos\vartheta)\,\times \\
& \cos\vartheta\sin\vartheta\, d\vartheta\,[1 + \lim_{\lambda\to 0} I_{z1}(\lambda)] \\
I_{z1}(\lambda) = {} & \int_\lambda^1 (gf_m + fg_m - 2gw_m)\lambda_1^2\, d\lambda_1
\end{aligned}
\tag{6}
$$

To select the perturbations that define the momentum I_z it is necessary to set $k = 0$, since it is only then that the integral in variable φ is nonzero. The constant b_{lk} may, without loss of generality, be set equal to zero. Let us consider the integral in ϑ; for $k = 0$ the associated Legendre functions become Legendre polynomials $P_l(\cos\vartheta)$. Taking into account that $P_1(\cos\vartheta) = \cos\vartheta$ and, also, the orthogonality of Legendre polynomials, we conclude that the integral in ϑ is nonzero only for $l = 1$. We select m so that the expression for I_z is independent of time. This yields $m = {}^3/_2$.

Note that for $l = 1$ and $m = {}^3/_2$ the system of Eqs. (5) admits the momentum integral

$$\lambda\,(g f_{3/2} + f g_{3/2} - 2g w_{3/2}) - [4fg f_{3/2} + 2f^2 g_{3/2} - 4fg w_{3/2} + (\varkappa - 1)\,h_{3/2}]\,/\,(\varkappa + 1) = C_{\mathrm{I}}\,/\,\lambda^2 \tag{7}$$

Using conditions (4) we obtain that the constant $C_{\mathrm{I}} = 0$. Eliminating in the system of Eqs. (5) function $w_{3/2}$ using equality (7) and discarding the third of Eqs. (5), we obtain a system of three equations for the determinations of functions $f_{3/2}$, $g_{3/2}$, and



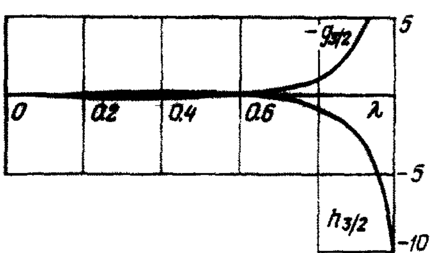


Fig. 1

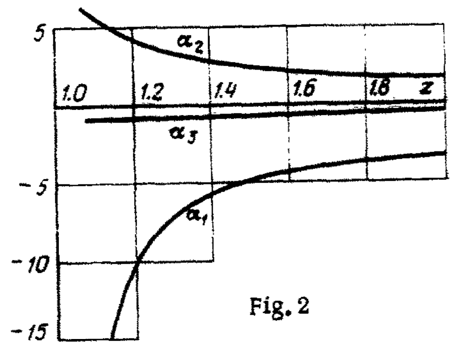


Fig. 2

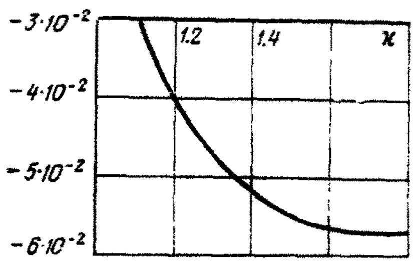


Fig. 3

$h_{3/2}$. This system was solved numerically for $\varkappa = 1.4$. Curves of these functions are shown in Fig. 1. For $\lambda \to 0$ all functions are of oscillatory character, but while $g_{3/2}$ and $h_{3/2}$ tend to zero $f_{3/2}$ and $w_{3/2}$ increase indefinitely; however, in spite of this the integral $I_{z\mathrm{I}}(\lambda)$ tends to a finite value equal -1.0521. Hence the motion with momentum

$$I_z = -\,0.1389\, C_0 \rho_1 b^{5/2} \qquad (C_0 > 0)$$

directed along the z-axis corresponds to the perturbation of a centrally symmetric shock front (1) with $t^{-2m/5} R\,(\varphi, \vartheta) = C_0 t^{-3/5} \cos\vartheta$. Let us prove the convergence of integral $I_{z\mathrm{I}}(\lambda)$ when $\lambda \to 0$. For this we analyze the asymptotics of second approximation functions for $\lambda \to 0$ and any arbitrary $\varkappa$ $(1 < \varkappa < 2)$. Since these functions satisfy the system of third order differential equations, the complete solution of that system consists of the sum of three linearly independent solutions.

The asymptotics of two linearly independent solutions are of an oscilatory character

$$f_{3/2} = \frac{(\varkappa^2-1)(\varkappa-1)}{6\varkappa k_1}\left\{\left[\alpha_1 + \frac{\varkappa+2}{2(\varkappa-1)}\right]C_{20}(\lambda) + \alpha_2 C_{30}(\lambda)\right\}\lambda^{\alpha_1} + \ldots \tag{8}$$

$$w_{3/2} = -\frac{(\varkappa^2-1)(\varkappa-1)}{12\varkappa k_1}\left\{\left[\alpha_1^2 - \alpha_2^2 + \frac{5\varkappa-2}{2(\varkappa-1)}\alpha_1 + \frac{\varkappa+2}{\varkappa-1}\right]C_{20}(\lambda) + \alpha_2\left[2\alpha_1 + \frac{5\varkappa-2}{2(\varkappa-1)}\right]C_{30}(\lambda)\right\}\lambda^{\alpha_1} + \ldots$$

$$g_{3/2} = C_{20}(\lambda)\lambda^{\beta_1} + \ldots, \quad \beta_1 = \alpha_1 + (4-\varkappa)/(\varkappa-1)$$

$$h_{3/2} = \frac{(\varkappa^2-1)^2}{6\varkappa^2}\left\{\left[\alpha_1^2 - \alpha_2^2 + \frac{7\varkappa-6}{2(\varkappa-1)}\alpha_1 + \frac{3\varkappa^2+2\varkappa-2}{2(\varkappa-1)^2}\right]C_{20}(\lambda) + \alpha_2\left[2\alpha_1 + \frac{7\varkappa-6}{2(\varkappa-1)}\right]C_{30}(\lambda)\right\}\lambda^{\gamma_1} + \ldots, \quad \gamma_1 = \alpha_1 + \frac{(2+\varkappa)}{(\varkappa-1)}$$

$$C_{20}(\lambda) = C_2\cos(\alpha_2 \ln\lambda) + C_3 \sin(\alpha_2\ln\lambda), \quad C_{30}(\lambda) = -C_2\sin(\alpha_2\ln\lambda) + C_3\cos(\alpha_2\ln\lambda)$$

while the third linearly independent solution is of the power function kind

$$f_{3/2} = C_4\frac{(\varkappa^2-1)(\varkappa-1)}{6\varkappa k_1}\left[\alpha_3 + \frac{\varkappa+2}{2(\varkappa-1)}\right]\lambda^{\alpha_3} + \ldots \tag{9}$$

$$w_{3/2} = -C_4\frac{(\varkappa^2-1)(\varkappa-1)}{12\varkappa k_1}\left[\alpha_3^2 + \frac{5\varkappa-2}{2(\varkappa-1)}\alpha_3 + \frac{\varkappa+2}{\varkappa-1}\right]\lambda^{\alpha_3} + \ldots$$

$$g_{3/2} = C_4\lambda^{\beta_3} + \ldots, \quad \beta_3 = \alpha_3 + (4-\varkappa)/(\varkappa-1)$$

$$h_{3/2} = C_4\frac{(\varkappa^2-1)^2}{6\varkappa^2}\left[\alpha_3^2 + \frac{7\varkappa-6}{2(\varkappa-1)}\alpha_3 + \frac{3\varkappa^2+2\varkappa-2}{2(\varkappa-1)^2}\right]\lambda^{\gamma_3} + \ldots$$

$$\gamma_3 = \alpha_3 + (2+\varkappa)/(\varkappa-1)$$

where C_2, C_3, and C_4 are arbitrary constants which in a specific problem are determined by the input data (4); k_1 is a coefficient in the expansion of density g for $\lambda \to 0$, and $\alpha_1 + i\alpha_2$, $\alpha_1 - i\alpha_2$, and α_3 (i is the imaginary unit) are roots of the cubic equation

$$\alpha^3 + \frac{7\varkappa}{2(\varkappa-1)}\alpha^2 + \frac{3\varkappa^2+17\varkappa-14}{2(\varkappa-1)^2}\alpha + \frac{3\varkappa}{(\varkappa-1)^2} = 0$$

Curves representing the dependence of α_1, α_2, and α_3 on $\varkappa$ are shown in Fig. 2. For $\varkappa = 1.4$ we have $\alpha_1 = -5.8101$, $\alpha_2 = 2.8150$, and $\alpha_3 = -0.6298$. These curves show that $\alpha_1 < \alpha_3$ in the whole range of $\varkappa$ variation, hence asymptotics (8) are higher than asymptotics (9), and the second approximation functions for $\lambda \to 0$ are always of an oscillating character of period $L = (e^{2\pi/\alpha_2} - 1)\lambda$ which vanishes together with the variable λ.

Asymptotics (8) imply that the integrand in formula (6) tends to zero when $\lambda \to 0$, which shows that the entire integral (6) is convergent, and that $I_{z1}(0)$, exists for any $\varkappa$, hence a motion with momentum

$$I_z = \frac{16 C_0 \rho_1 b^{3/2}}{15(\varkappa-1)}[1 + I_{z1}(0)]$$

directed along the z-axis corresponds to perturbation (1) of the form $C_0 t^{-3/4} \cos \vartheta$.

The dependence of $1 + I_{z1}(0)$ on $\varkappa$ is shown in Fig. 3. Since $1 + I_{z1}(0) < 0$, hence the direction of momentum I_z is opposite to the relative shift of the shock front along the z-axis which is defined by the quantity $C_0 t^{-3/4}$. This situation is not unusual: the phenomenon of irregular shock wave shift occurs in asymmetric hypersonic flow past a circular cone [13].

We point out a further importnat property of the derived solution, namely, that for $\lambda \to 0$ the perturbed pressure in conformity with (7) also tends to zero. This makes it possible to continue the solution in the region close to the center, where pressure is almost invariant with respect to space variables. Such continuation can be effected with allowance for viscosity and heat conduction, as was done in [9] for plane-parallel motions with constant energy and momentum.

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